HIGHER ORDER ASYMPTOTIC FORMULAS FOR TRACES OF TOEPLITZ MATRICES WITH SYMBOLS IN HÖLDER-ZYGMUND SPACES

ALEXEI YU. KARLOVICH

Abstract. We prove a higher order asymptotic formula for traces of finite block Toeplitz matrices with symbols belonging to Hölder-Zygmund spaces. The remainder in this formula goes to zero very rapidly for very smooth symbols. This formula refine previous asymptotic trace formulas by Szegő and Widom and complement higher order asymptotic formulas for determinants of finite block Toeplitz matrices due to Böttcher and Silbermann.

1. Introduction and main result

1.1. **Finite block Toeplitz matrices.** Let $\mathbb{Z},\mathbb{N},\mathbb{Z}_+$, and $\mathbb{C}$ be the sets of integers, positive integers, nonnegative integers, and all complex numbers, respectively. Suppose $N\in\mathbb{N}$. For a Banach space X, let X_N and $X_{N\times N}$ be the spaces of vectors and matrices with entries in X. Let $\mathbb{T}$ be the unit circle. For $1\le p\le\infty$, let $L^p:=L^p(\mathbb{T})$ and $H^p:=H^p(\mathbb{T})$ be the standard Lebesgue and Hardy spaces of the unit circle. For $a\in L^1_{N\times N}$ one can define

$$a_k=\frac{1}{2\pi}\int_0^{2\pi}a(e^{i\theta})e^{-ik\theta}d\theta\quad(k\in\mathbb{Z}),$$

the sequence of the Fourier coefficients of a. Let I be the identity operator, P be the Riesz projection of L^2 onto H^2, $Q:=I-P$, and define I,P, and Q on L^2_N elementwise. For $a\in L^\infty_{N\times N}$ and $t\in\mathbb{T}$, put $\widetilde{a}(t):=a(1/t)$ and $(Ja)(t):=t^{-1}\widetilde{a}(t)$. Define *Toeplitz operators*

$$T(a):=PaP|\operatorname{Im}P,\quad T(\widetilde{a}):=JQaQJ|\operatorname{Im}P$$

and *Hankel operators*

$$H(a):=PaQJ|\operatorname{Im}P,\quad H(\widetilde{a}):=JQaP|\operatorname{Im}P.$$

The function a is called the *symbol* of $T(a)$, $T(\widetilde{a})$, $H(a)$, $H(\widetilde{a})$. We are interested in the asymptotic behavior of *finite block Toeplitz matrices* $T_n(a)=[a_{j-k}]_{j,k=0}^n$ generated by (the Fourier coefficients of) the symbol a as $n\to\infty$. Many results in this direction are contained in the books by Grenander and Szegő [10], Böttcher and Silbermann [3, 4, 5], Simon [18], and Böttcher and Grudsky [1].

2000 *Mathematics Subject Classification.* Primary 47B35; Secondary 15A15, 47B10, 47L20, 47A68.

Key words and phrases. Block Toeplitz matrix, determinant, trace, strong Szegő-Widom theorem, decomposing algebra, canonical Wiener-Hopf factorization, Hölder-Zygmund space.

This work is supported by Centro de Matemática da Universidade do Minho (Portugal) and by the Portuguese Foundation of Science and Technology through the research program POCTI.

1.2. **Szegő-Widom limit theorems.** Let us formulate precisely the most relevant results. Let $K^2_{N\times N}$ be the Krein algebra [12] of matrix functions a in $L^\infty_{N\times N}$ satisfying

$$\sum_{k=-\infty}^{\infty} \|a_k\|^2(|k|+1) < \infty,$$

where $\|\cdot\|$ is any matrix norm on $\mathbb{C}_{N\times N}$. The following beautiful theorem about the asymptotics of finite block Toeplitz matrices was proved by Widom [21].

Theorem 1.1. (see [21, Theorem 6.1]). *If $a \in K^2_{N\times N}$ and the Toeplitz operators $T(a)$ and $T(\widetilde{a})$ are invertible on H^2_N, then $T(a)T(a^{-1}) - I$ is of trace class and, with appropriate branches of the logarithm,*

$$\log\det T_n(a) = (n+1)\log G(a) + \log\det T(a)T(a^{-1}) + o(1) \quad \textit{as} \quad n\to\infty, \tag{1}$$

where

$$G(a) := \lim_{r\to 1-0} \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \log\det \widehat{a}_r(e^{i\theta})d\theta\right), \widehat{a}_r(e^{i\theta}) := \sum_{n=-\infty}^{\infty} a_n r^{|n|} e^{in\theta}. \tag{2}$$

In formula (1), $\det T(a)T(a^{-1})$ refers to the determinant defined for operators on Hilbert space differing from the identity by an operator of trace class [9, Chap. 4].

The proof of the above result in a more general form is contained in [3, Theorem 6.11] and [5, Theorem 10.30] (in this connection see also [8]).

Let $\lambda_1^{(n)}, \dots, \lambda_{(n+1)N}^{(n)}$ denote the eigenvalues of $T_n(a)$ repeated according to their algebraic multiplicity. Let $\operatorname{sp} A$ denote the spectrum of a bounded linear operator A and $\operatorname{tr} M$ denote the trace of a matrix M. Theorem 1.1 is equivalent to the assertion

$$\sum_i \log\lambda_i^{(n)} = \operatorname{tr}\,\log T_n(a) = (n+1)\log G(a) + \log\det T(a)T(a^{-1}) + o(1).$$

Widom [21] noticed that Theorem 1.1 yields even a description of the asymptotic behavior of $\operatorname{tr} f(T_n(a))$ if one replaces $f(\lambda) = \log\lambda$ by an arbitrary function f analytic in an open neighborhood of the union $\operatorname{sp} T(a) \cup \operatorname{sp} T(\widetilde{a})$ (we henceforth call such f simply analytic on $\operatorname{sp} T(a) \cup \operatorname{sp} T(\widetilde{a})$).

Theorem 1.2. (see [21, Theorem 6.2]). *If $a \in K^2_{N\times N}$ and if f is analytic on $\operatorname{sp} T(a) \cup \operatorname{sp} T(\widetilde{a})$, then*

$$\operatorname{tr} f(T_n(a)) = (n+1)G_f(a) + E_f(a) + o(1) \quad \textit{as} \quad n\to\infty, \tag{3}$$

where

$$\begin{aligned} G_f(a) &:= \frac{1}{2\pi}\int_0^{2\pi} (\operatorname{tr} f(a))(e^{i\theta})d\theta, \\ E_f(a) &:= \frac{1}{2\pi i}\int_{\partial\Omega} f(\lambda)\frac{d}{d\lambda}\log\det T[a-\lambda]T[(a-\lambda)^{-1}]d\lambda, \end{aligned}$$

and Ω is any bounded open set containing $\operatorname{sp} T(a) \cup \operatorname{sp} T(\widetilde{a})$ on the closure of which f is analytic.

The proof of Theorem 1.2 for continuous symbols a is also given in [5, Section 10.90]. In the scalar case ($N = 1$) Theorems 1.1 and 1.2 go back to Gabor Szegő (see [10] and historical remarks in [3, 4, 5, 18]).

1.3. **Hölder-Zygmund spaces.** Suppose g is a bounded function on $\mathbb{T}$. The *modulus of continuity* of g is defined for $s \ge 0$ by

$$\omega_1(g,s) := \sup\left\{|g(e^{i(x+h)}) - g(e^{ix})| : \ x,h \in \mathbb{R},\ |h| \le s\right\}.$$

By the *modulus of smoothness* (of order 2) of g is meant the function (see, e.g., [19, Section 3.3]) defined for $s \ge 0$ by

$$\omega_2(g,s) := \sup\left\{|g(e^{i(x+h)}) - 2g(e^{ix}) + g(e^{i(x-h)})| : \ x,h \in \mathbb{R},\ |h| \le s\right\}.$$

Let $C = C(\mathbb{T})$ be the set of all continuous functions on $\mathbb{T}$. Given $\gamma > 0$, write $\gamma = m + \delta$, where $m \in \mathbb{Z}_+$ and $\delta \in (0,1]$. The Hölder-Zygmund space $C^\gamma = C^\gamma(\mathbb{T})$ is defined (see, e.g., [16, Section 3.5.4]) by

$$C^\gamma := \left\{f \in C : f^{(j)} \in C,\ 1 \le j \le m,\ [f^{(m)}]_\delta < \infty\right\}$$

with the norm

$$\|f\|_\gamma := \sum_{j=0}^{m} \|f^{(j)}\|_\infty + [f^{(m)}]_\delta,$$

where $f^{(j)}$ is the derivative of order j of f, $\|\cdot\|_\infty$ is the norm in L^∞, and

$$[g]_\delta := \sup_{s>0} \frac{\omega_2(g,s)}{s^\delta}, \quad 0 < \delta \le 1.$$

Notice that if $\gamma > 0$ is not integer, then $[g]_\delta$ can be replaced by

$$[g]_\delta^* := \sup_{s>0} \frac{\omega_1(g,s)}{s^\delta}, \quad 0 < \delta < 1$$

in the above definition.

1.4. **Böttcher-Silbermann higher order asymptotic formulas for determinants.** Following [21] and [5, Sections 7.5–7.6], for $n \in \mathbb{Z}_+$ and $a \in L^\infty_{N\times N}$ define the operators P_n and Q_n on H^2_N by

$$P_n : \sum_{k=0}^{\infty} a_k t^k \mapsto \sum_{k=0}^{n} a_k t^k, \quad Q_n := I - P_n.$$

The operator $P_n T(a) P_n : P_n H^2_N \to P_n H^2_N$ may be identified with the finite block Toeplitz matrix $T_n(a) := [a_{j-k}]_{j,k=0}^n$. For a unital Banach algebra A we will denote by GA the group of all invertible elements of A. For $1 \le p \le \infty$, put

$$H^p_\pm \ := \ \left\{a \in L^p : a_{\mp n} = 0 \text{ for } n \in \mathbb{N}\right\}.$$

Böttcher and Silbermann [2] proved among other things the following result.

Theorem 1.3. *Let $p \in \mathbb{N}$ and $\alpha, \beta > 0$ satisfy $\alpha + \beta > 1/p$. Suppose $a = u_- u_+$, where $u_+ \in G(C^\alpha \cap H^\infty_+)_{N\times N}$ and $u_- \in G(C^\beta \cap H^\infty_-)_{N\times N}$, and the Toeplitz operator $T(\widetilde{a})$ is invertible on H^2_N. Then*

(a) *there exist $v_- \in G(H^\infty_-)_{N\times N}$ and $v_+ \in G(H^\infty_+)_{N\times N}$ such that $a = v_+ v_-$;*

(b) *there exist a constant $\widetilde{E}(a) \ne 0$ such that*

$$\begin{aligned}\log\det T_n(a) \ &= \ (n+1)\log G(a) + \log \widetilde{E}(a) \\ &\quad + \operatorname{tr}\left[\sum_{\ell=1}^{n}\sum_{j=1}^{p-1}\frac{1}{j}\left(\sum_{k=0}^{p-j-1} G_{\ell,k}(b,c)\right)^j\right] \\ &\quad + O(1/n^{(\alpha+\beta)p-1})\end{aligned}$$

as $n \to \infty$, where the correcting terms $G_{\ell,k}(b,c)$ are given by

$$G_{\ell,k}(b,c) := P_0 T(c) Q_\ell \big(Q_\ell H(b) H(\widetilde{c}) Q_\ell\big)^k Q_\ell T(b) P_0 \quad (\ell, k \in \mathbb{Z}_+) \tag{4}$$

and the functions b, c are given by $b := v_- u_+^{-1}$ and $c := u_-^{-1} v_+$.

If, in addition, $p = 1$, then

(c) *the operator $T(a)T(a^{-1}) - I$ is of trace class and*

$$\log\det T_n(a) = (n+1)\log G(a) + \log\det T(a)T(a^{-1}) + O(1/n^{\alpha+\beta-1}) \tag{5}$$

as $n \to \infty$.

The sketch of the proof of parts (a) and (b) is contained in [3, Sections 6.18(ii)] and in [5, Theorem 10.35(ii)]. Part (c) is explicitly stated in [3, Section 6.18(ii)] or immediately follows from [5, Theorems 10.35(ii) and 10.37(ii)].

1.5. **Our main result.** Our main result is the following refinement of Theorem 1.2.

Theorem 1.4. *Let $\gamma > 1/2$. If $a \in C^\gamma_{N\times N}$ and if f is analytic on $\operatorname{sp} T(a) \cup \operatorname{sp} T(\widetilde{a})$, then (3) is true with $o(1)$ replaced by $O(1/n^{2\gamma-1})$.*

Clearly, this result is predicted by Theorem 1.3(c) with $\gamma = \alpha = \beta$. The key point in the Widom's proof of Theorem 1.2 is that (1) is valid for $a - \lambda$ in place of a, uniformly with respect to λ in a neighborhood of $\partial\Omega$. We will show that the same remains true for the higher order asymptotic formula (5) with $\gamma = \alpha = \beta$. In Section 2 we collect necessary information about right and left Wiener-Hopf factorizations in decomposing algebras and mention that a nonsingular matrix function belonging to a Hölder-Zygmund space $C^\gamma_{N\times N}$ ($\gamma > 0$) admits right and left Wiener-Hopf factorizations in $C^\gamma_{N\times N}$. In Section 3 we give the proof of Theorem 1.4 using an idea of Böttcher and Silbermann [2] of a decomposition of $\operatorname{tr}\log\{I - \sum_{k=0}^\infty G_{n,k}(b,c)\}$. We show that this decomposition can be made for $a - \lambda$ uniform with respect to λ in a neighborhood of $\partial\Omega$. This actually implies that (5) is valid with $\gamma = \alpha = \beta$ and a replaced by $a - \lambda$ uniformly with respect to λ in a neighborhood of $\partial\Omega$. Thus, Widom's arguments apply.

1.6. **Higher order asymptotic trace formulas for Toeplitz matrices with symbols from other smoothness classes.** Let us mention two other classes of symbols for which higher order asymptotic formulas for $\operatorname{tr} f(T_n(a))$ are available.

Theorem 1.5. *Suppose a is a continuous $N \times N$ matrix function on the unit circle and f is analytic on $\operatorname{sp} T(a) \cup \operatorname{sp} T(\widetilde{a})$. Let $\|\cdot\|$ be any matrix norm on $\mathbb{C}_{N\times N}$.*

(a) (see [20]). *If $\gamma > 1$ and*

$$\sum_{k=-\infty}^{\infty} \|a_k\| + \sum_{k=-\infty}^{\infty} \|a_k\|^2 |k|^\gamma < \infty,$$

then (3) is true with $o(1)$ replaced by $o(1/n^{\gamma-1})$.

(b) (see [11, Corollary 1.6]). *If $\alpha, \beta > 0$, $\alpha + \beta > 1$, and*

$$\sum_{k=1}^{\infty} \|a_{-k}\| k^\alpha + \sum_{k=1}^{\infty} \|a_k\| k^\beta < \infty,$$

then (3) is true with $o(1)$ replaced by $o(1/n^{\alpha+\beta-1})$.

2. Wiener-Hopf factorization in decomposing algebras of continuous functions

2.1. **Definitions and general theorems.** Let $\mathbb{D}$ be the open unit disk. Let $\mathcal{R}_-$ (resp. $\mathcal{R}_+$) denote the set of all rational functions with poles only in $\mathbb{D}$ (resp. in $(\mathbb{C} \cup \{\infty\}) \setminus (\mathbb{D} \cup \mathbb{T})$). Let $C_\pm$ be the closure of $\mathcal{R}_\pm$ with respect to the norm of C. Suppose $\mathcal{A}$ is a Banach algebra of continuous functions on $\mathbb{T}$ that contains $\mathcal{R}_+ \cup \mathcal{R}_-$ and has the following property: if $a \in \mathcal{A}$ and $a(t) \neq 0$ for all $t \in \mathbb{T}$, then $a^{-1} \in \mathcal{A}$. The sets $\mathcal{A}_\pm := \mathcal{A} \cap C_\pm$ are subalgebras of $\mathcal{A}$. The algebra $\mathcal{A}$ is said to be *decomposing* if every function $a \in \mathcal{A}$ can be represented in the form $a = a_- + a_+$ where $a_\pm \in \mathcal{A}_\pm$.

Let $\mathcal{A}$ be a decomposing algebra. A matrix function $a \in \mathcal{A}_{N\times N}$ is said to admit a *right* (resp. *left*) *Wiener-Hopf* (WH) *factorization in* $\mathcal{A}_{N\times N}$ if it can be represented in the form $a = a_- d a_+$ (resp. $a = a_+ d a_-$), where

$$a_\pm \in G(\mathcal{A}_\pm)_{N\times N}, \quad d(t) = \operatorname{diag}\{t^{\kappa_1}, \dots, t^{\kappa_N}\}, \quad \kappa_i \in \mathbb{Z}, \quad \kappa_1 \le \dots \le \kappa_N.$$

The integers κ_i are usually called the *right* (resp. *left*) *partial indices* of a; they can be shown to be uniquely determined by a. If $\kappa_1 = \dots = \kappa_N = 0$, then the respective WH factorization is said to be *canonical*.

The following result was obtained by Budjanu and Gohberg [6, Theorem 4.3] and it is contained in [7, Chap. II, Corollary 5.1] and in [13, Theorem 5.7'].

Theorem 2.1. *Suppose the following two conditions hold for the algebra $\mathcal{A}$:*

(a) *the Cauchy singular integral operator*

$$(S\varphi)(t) := \frac{1}{\pi i} v.p. \int_{\mathbb{T}} \frac{\varphi(\tau)}{\tau - t} d\tau \quad (t \in \mathbb{T})$$

is bounded on $\mathcal{A}$;

(b) *for any function $a \in \mathcal{A}$, the operator $aS - SaI$ is compact on $\mathcal{A}$.*

Then every matrix function $a \in \mathcal{A}_{N\times N}$ such that $\det a(t) \neq 0$ *for all $t \in \mathbb{T}$ admits a right and left WH factorizations in $\mathcal{A}_{N\times N}$ (in general, with different sets of partial indices).*

Notice that (a) holds if and only if $\mathcal{A}$ is a decomposing algebra.

The following theorem follows from a more general result due to Shubin [17]. Its proof can be found in [13, Theorem 6.15].

Theorem 2.2. *Let $\mathcal{A}$ be a decomposing algebra and let $\|\cdot\|$ be a norm in the algebra $\mathcal{A}_{N\times N}$. Suppose $a, c \in \mathcal{A}_{N\times N}$ admit canonical right and left WH factorizations in the algebra $\mathcal{A}_{N\times N}$. Then for every $\varepsilon > 0$ there exists a $\delta > 0$ such that if $\|a-c\| < \delta$, then for every canonical right WH factorization $a = a_-^{(r)} a_+^{(r)}$ and for every canonical left WH factorization $a = a_+^{(l)} a_-^{(l)}$ one can choose a canonical right WH factorization $c = c_-^{(r)} c_+^{(r)}$ and a canonical left WH factorization $c = c_+^{(l)} c_-^{(l)}$ such that*

$$\|a_\pm^{(r)} - c_\pm^{(r)}\| < \varepsilon, \qquad \|[a_\pm^{(r)}]^{-1} - [c_\pm^{(r)}]^{-1}\| < \varepsilon,$$

$$\|a_\pm^{(l)} - c_\pm^{(l)}\| < \varepsilon, \qquad \|[a_\pm^{(l)}]^{-1} - [c_\pm^{(l)}]^{-1}\| < \varepsilon.$$

2.2. **Wiener-Hopf factorization in Hölder-Zygmund spaces.**

Theorem 2.3. (see [15, Section 6.25]). *Suppose $\gamma > 0$. Then*

(a) *C^γ is a Banach algebra;*

(b) *$a \in C^\gamma$ is invertible in C^γ if and only if $a(t) \neq 0$ for all $t \in \mathbb{T}$;*
(c) *S is bounded on C^γ;*
(d) *for $a \in C^\gamma$, the operator aI is bounded on C^γ and the operator $aS - SaI$ is compact on C^γ.*

For $\gamma \notin \mathbb{Z}_+$, parts (c) and (d) are proved in [6, Section 7] (see also [7, Chap. II, Section 6.2]). Note that a statement similar to (d) is proved in [14, Chap. 7, Theorem 4.3].

Theorem 2.4. *Let $\gamma > 0$ and Σ be a compact set in the complex plane. Suppose $a : \Sigma \to C^\gamma_{N\times N}$ is a continuous function and the Toeplitz operators $T(a(\lambda))$ and $T([a(\lambda)]^\sim)$ are invertible on H^2_N for all $\lambda \in \Sigma$. Then for every $\lambda \in \Sigma$ the function $a(\lambda) : \mathbb{T} \to \mathbb{C}$ admits canonical right and left WH factorizations*

$$a(\lambda) = u_-(\lambda)u_+(\lambda) = v_+(\lambda)v_-(\lambda)$$

in $C^\gamma_{N\times N}$. These factorizations can be chosen so that $u_\pm, v_\pm, u_\pm^{-1}, v_\pm^{-1} : \Sigma \to C^\gamma_{N\times N}$ are continuous.

Proof. Fix $\lambda \in \Sigma$ and put $a := a(\lambda)$. If $T(a)$ is invertible on H^2_N, then $\det a(t) \neq 0$ for all $t \in \mathbb{T}$ (see, e.g., [7, Chap. VII, Proposition 2.1]). Then, by [7, Chap. VII, Theorem 3.2], the matrix function a admits a canonical right generalized factorization in L^2_N, that is, $a = a_-a_+$, where $a_-^{\pm1} \in (H^2_-)_{N\times N}$, $a_+^{\pm1} \in (H^2_+)_{N\times N}$ (and, moreover, the operator $a_-Pa_-^{-1}I$ is bounded on L^2_N).

On the other hand, from Theorems 2.1 and 2.3 it follows that $a \in C^\gamma_{N\times N}$ admits a right WH factorization $a = u_-du_+$ in $C^\gamma_{N\times N}$. Then

$$u_\pm \in (C^\gamma_\pm)_{N\times N} \subset (H^2_\pm)_{N\times N}, \quad u_\pm^{-1} \in (C^\gamma_\pm)_{N\times N} \subset (H^2_\pm)_{N\times N}.$$

By the uniqueness of the partial indices in a right generalized factorization in L^2_N (see, e.g., [13, Corollary 2.1]), $d = 1$.

Let us prove that a admits also a canonical left WH factorization in the algebra $C^\gamma_{N\times N}$. In view of Theorem 2.3(b), $a^{-1} \in C^\gamma_{N\times N}$. By [5, Proposition 7.19(b)], the invertibility of $T(\widetilde{a})$ on H^2_N is equivalent to the invertibility of $T(a^{-1})$ on H^2_N. By what has just been proved, there exist $f_\pm \in G(C^\gamma_\pm)_{N\times N}$ such that $a^{-1} = f_-f_+$. Put $v_\pm := f_\pm^{-1}$. Then $v_\pm \in G(C^\gamma_\pm)_{N\times N}$ and $a = v_+v_-$ is a canonical left WH factorization in $C^\gamma_{N\times N}$.

We have proved that for each $\lambda \in \Sigma$ the matrix function $a(\lambda) : \mathbb{T} \to \mathbb{C}$ admits canonical right and left WH factorizations in $C^\gamma_{N\times N}$. By Theorem 2.2, these factorizations can be chosen so that the factors $u_\pm, v_\pm$ and their inverses $u_\pm^{-1}, v_\pm^{-1}$ are continuous functions from Σ to $C^\gamma_{N\times N}$. □

3. Proof of the main result

3.1. **The Böttcher-Silbermann decomposition.** The following result from [3, Section 6.16], [5, Section 10.34] is the basis for our asymptotic analysis.

Lemma 3.1. *Suppose $a \in L^\infty_{N\times N}$ satisfies the following hypotheses:*

(i) *there are two factorizations $a = u_-u_+ = v_+v_-$, where $u_+, v_+ \in G(H^\infty_+)_{N\times N}$ and $u_-, v_- \in G(H^\infty_-)_{N\times N}$;*
(ii) *$u_- \in C_{N\times N}$ or $u_+ \in C_{N\times N}$.*

Define the functions b, c by $b := v_- u_+^{-1}$, $c := u_-^{-1} v_+$ and the matrices $G_{n,k}(b,c)$ by (4). Suppose for all sufficiently large n (say, $n \ge N_0$) there exists a decomposition

$$\operatorname{tr}\,\log\left\{I - \sum_{k=0}^{\infty} G_{n,k}(b,c)\right\} = -\operatorname{tr} H_n + s_n \tag{6}$$

where $\{H_n\}_{n=N_0}^{\infty}$ is a sequence of $N \times N$ matrices and $\{s_n\}_{n=N_0}^{\infty}$ is a sequence of complex numbers. If $\sum_{n=N_0}^{\infty} |s_n| < \infty$, then there exist a constant $\widetilde{E}(a) \ne 0$ depending on $\{H_n\}_{n=N_0}^{\infty}$ and arbitrarily chosen $N \times N$ matrices $H_1, \dots, H_{N_0-1}$ such that for all $n \ge N_0$,

$$\log\det T_n(a) = (n+1)\log G(a) + \operatorname{tr}(H_1 + \dots + H_n) + \log \widetilde{E}(a) + \sum_{k=n+1}^{\infty} s_k,$$

where the constant $G(a)$ is given by (2).

3.2. **The best uniform approximation.** Let $\mathcal{P}^n$ be the set of all Laurent polynomials of the form

$$p(t) = \sum_{j=-n}^{n} \alpha_j t^j, \quad \alpha_j \in \mathbb{C}, \quad t \in \mathbb{T}.$$

By the Chebyshev theorem (see, e.g., [19, Section 2.2.1]), for $f \in C$ and $n \in \mathbb{N}$, there is a Laurent polynomial $p_n(f) \in \mathcal{P}^n$ such that

$$\|f - p_n(f)\|_\infty = \inf_{p \in \mathcal{P}^n} \|f - p\|_\infty. \tag{7}$$

This polynomial $p_n(f)$ is called a polynomial of best uniform approximation.

By the Jackson-Ahiezer-Stechkin theorem (see, e.g., [19, Section 5.1.4]), if f has a bounded derivative $f^{(m)}$ of order m on $\mathbb{T}$, then for $n \in \mathbb{N}$,

$$\inf_{p \in \mathcal{P}^n} \|f - p\|_\infty \le \frac{C_m}{(n+1)^m}\omega_2\left(f^{(m)}, \frac{1}{n+1}\right), \tag{8}$$

where the constant C_m depends only on m.

From (7) and (8) it follows that if $f \in C^\gamma$ and $n \in \mathbb{N}$, where $\gamma = m + \delta$ with $m \in \mathbb{Z}_+$ and $\delta \in (0,1]$, then there is a $p_n(f) \in \mathcal{P}^n$ such that

$$\|f - p_n(f)\|_\infty \le \frac{C_m}{(n+1)^m}\omega_2\left(f^{(m)}, \frac{1}{n+1}\right) \le \frac{C_m [f^{(m)}]_\delta}{(n+1)^{m+\delta}} \le C_m \frac{\|f\|_\gamma}{n^\gamma}. \tag{9}$$

3.3. **Norms of truncations of Toeplitz and Hankel operators.** Let X be a Banach space. For definiteness, let the norm of $a = [a_{ij}]_{i,j=1}^N$ in $X_{N\times N}$ be given by $\|a\|_{X_{N\times N}} = \max_{1 \le i,j \le N} \|a_{ij}\|_X$. We will simply write $\|a\|_\infty$ and $\|a\|_\gamma$ instead of $\|a\|_{L^\infty_{N\times N}}$ and $\|a\|_{C^\gamma_{N\times N}}$, respectively. Denote by $\|A\|$ the norm of a bounded linear operator A on H^2_N.

A slightly less precise version of the following statement was used in the proof of [5, Theorem 10.35(ii)].

Proposition 3.2. *Let $\alpha, \beta > 0$. Suppose $b = v_- u_+^{-1}$ and $c = u_-^{-1} v_+$, where*

$$u_+ \in G(C^\alpha \cap H^\infty_+)_{N\times N}, \quad u_- \in G(C^\beta \cap H^\infty_-)_{N\times N}, \quad v_\pm \in G(H^\infty_\pm)_{N\times N}.$$

Then there exist positive constants M_α and M_β depending only on N and α and β, respectively, such that for all $n \in \mathbb{N}$,

$$\|Q_n T(b) P_0\| \le \frac{M_\alpha}{n^\alpha}\|v_-\|_\infty \|u_+^{-1}\|_\alpha, \quad \|Q_n H(b)\| \le \frac{M_\alpha}{n^\alpha}\|v_-\|_\infty \|u_+^{-1}\|_\alpha,$$

$$\|P_0 T(c) Q_n\| \le \frac{M_\beta}{n^\beta}\|v_+\|_\infty \|u_-^{-1}\|_\beta, \quad \|H(\widetilde{c}) Q_n\| \le \frac{M_\beta}{n^\beta}\|v_+\|_\infty \|u_-^{-1}\|_\beta.$$

Proof. Since $b = v_- u_+^{-1}$, $c = u_-^{-1} v_+$ and $v_\pm, u_\pm \in G(H^\infty_\pm)_{N\times N}$, one has

$$Q_n T(b) P_0 = Q_n T(v_-) Q_n T(u_+^{-1}) P_0, \tag{10}$$

$$Q_n H(b) = Q_n T(v_-) Q_n H(u_+^{-1}), \tag{11}$$

$$P_0 T(c) Q_n = P_0 T(u_-^{-1}) Q_n T(v_+) Q_n, \tag{12}$$

$$H(\widetilde{c}) Q_n = H(\widetilde{u_-^{-1}}) Q_n T(v_+) Q_n. \tag{13}$$

Let $p_n(u_+^{-1})$ and $p_n(u_-^{-1})$ be the polynomials in $\mathcal{P}^n_{N\times N}$ of best uniform approximation of u_+^{-1} and u_-^{-1}, respectively. Obviously,

$$Q_n T[p_n(u_+^{-1})] P_0 = 0, \qquad Q_n H[p_n(u_+^{-1})] = 0,$$

$$P_0 T[p_n(u_-^{-1})] Q_n = 0, \qquad H[(p_n(u_-^{-1}))\widetilde{\ }] Q_n = 0.$$

Then from (9) it follows that

$$\begin{aligned} \|Q_n T(u_+^{-1}) P_0\| &= \|Q_n T[u_+^{-1} - p_n(u_+^{-1})] P_0\| \\ &\le \|P\|\,\|u_+^{-1} - p_n(u_+^{-1})\|_\infty \le \frac{M_\alpha}{n_\alpha}\|u_+^{-1}\|_\alpha \end{aligned} \tag{14}$$

and similarly

$$\|Q_n H(u_+^{-1})\| \le \frac{M_\alpha}{n^\alpha}\|u_+^{-1}\|_\alpha, \tag{15}$$

$$\|P_0 T(u_-^{-1}) Q_n\| \le \frac{M_\beta}{n^\beta}\|u_-^{-1}\|_\beta, \tag{16}$$

$$\|H(\widetilde{u_-^{-1}}) Q_n\| \le \frac{M_\beta}{n^\beta}\|u_-^{-1}\|_\beta, \tag{17}$$

where M_α and M_β depend only on α, β and N. Combining (10) and (14), we get

$$\|Q_n T(b) P_0\| \le \|T(v_-)\|\,\|Q_n T(u_-^{-1}) P_0\| \le \frac{M_\alpha}{n^\alpha}\|v_-\|_\infty \|u_+^{-1}\|_\alpha.$$

All other assertions follow from (11)–(13) and (15)–(17). $\square$

3.4. **The key estimate.** The following proposition shows that a decomposition of Lemma 3.1 exists.

Proposition 3.3. *Suppose the conditions of Proposition 3.2 are fulfilled. If $p \in \mathbb{N}$, then there exists a constant $C_p \in (0,\infty)$ depending only on p such that*

$$\left| \mathrm{tr}\,\log\left\{ I - \sum_{k=0}^{\infty} G_{n,k}(b,c) \right\} + \mathrm{tr}\left[\sum_{j=1}^{p-1} \frac{1}{j} \left(\sum_{k=0}^{p-j-1} G_{n,k}(b,c) \right)^j \right] \right|$$
$$\le C_p \left(\frac{M_\alpha M_\beta}{n^{\alpha+\beta}} \|u_+^{-1}\|_\alpha \|u_-^{-1}\|_\beta \|v_-\|_\infty \|v_+\|_\infty \right)^p$$

for all $n > \left(M_\alpha M_\beta \|u_+^{-1}\|_\alpha \|u_-^{-1}\|_\beta \|v_-\|_\infty \|v_+\|_\infty\right)^{1/(\alpha+\beta)}$.

Proof. From Proposition 3.2 it follows that

$$\|G_{n,k}(b,c)\| \le \left[\frac{M_\alpha M_\beta}{n^{\alpha+\beta}}\|u_+^{-1}\|_\alpha \|u_-^{-1}\|_\beta \|v_-\|_\infty \|v_+\|_\infty\right]^{k+1}$$

for all $k \in \mathbb{Z}_+$ and $n \in \mathbb{N}$. If $n > \left(M_\alpha M_\beta \|u_+^{-1}\|_\alpha \|u_-^{-1}\|_\beta \|v_-\|_\infty \|v_+\|_\infty\right)^{1/(\alpha+\beta)}$, then the expression in the brackets is less than 1. In view of these observations the proof can be developed as in [11, Proposition 3.3]. □

Theorem 1.3(b) follows from the above statement and Lemma 3.1. In the next section we will use the partial case $p = 1$ of Proposition 3.3 as the key ingredient of the proof of our main result.

3.5. **Proof of Theorem 1.4.** Suppose $\gamma > 1/2$ and $\lambda \notin \operatorname{sp} T(a) \cup \operatorname{sp} T(\widetilde{a})$. Then

$$T(a) - \lambda I = T(a-\lambda), \quad T(\widetilde{a}) - \lambda I = T([a-\lambda]^{\sim})$$

are invertible on H^2_N. Since $a - \lambda$ is continuous with respect to λ as a function from a closed neighborhood Σ of $\partial\Omega$ to $C^\gamma_{N\times N}$, in view of Theorem 2.4, for each $\lambda \in \Sigma$, the function $a - \lambda : \mathbb{T} \to \mathbb{C}$ admits canonical right and left WH factorizations $a - \lambda = u_-(\lambda)u_+(\lambda) = v_+(\lambda)v_-(\lambda)$ in $C^\gamma_{N\times N}$ and these factorizations can be chosen so that the factors $u_\pm$, $v_\pm$ and their inverses $u_\pm^{-1}$, $v_\pm^{-1}$ are continuous from Σ to $C^\gamma_{N\times N}$. Then

$$A_\Sigma := \max_{\lambda\in\Sigma}\left(\|u_+^{-1}(\lambda)\|_\gamma \|u_-^{-1}(\lambda)\|_\gamma \|v_-(\lambda)\|_\gamma \|v_+(\lambda)\|_\gamma\right) < \infty.$$

Put $b = v_- u_+^{-1}$ and $c = u_-^{-1} v_+$. From Proposition 3.3 with $p = 1$ it follows that there exists $C_1 \in (0,\infty)$ such that

$$\begin{aligned} &\left|\operatorname{tr}\,\log\left\{I - \sum_{k=0}^{\infty} G_{n,k}(b(\lambda), c(\lambda))\right\}\right| \\ &\le \frac{C_1 M_\gamma^2}{n^{2\gamma}}\|u_+^{-1}(\lambda)\|_\gamma \|u_-^{-1}(\lambda)\|_\gamma \|v_-(\lambda)\|_\infty \|v_+(\lambda)\|_\infty \\ &\le \frac{C_1 M_\gamma^2 A_\Sigma}{n^{2\gamma}} \end{aligned} \tag{18}$$

for all $n > (M_\gamma^2 A_\Sigma)^{1/(2\gamma)}$ and all $\lambda \in \Sigma$. Obviously

$$\sum_{k=n+1}^{\infty} \frac{1}{k^{2\gamma}} = O(1/n^{2\gamma-1}). \tag{19}$$

From Lemma 3.1 and (18)–(19) it follows that there is a function $\widetilde{E}(a,\cdot) : \Sigma \to \mathbb{C} \setminus \{0\}$ such that

$$\log\det T_n(a-\lambda) = (n+1)\log G(a-\lambda) + \log \widetilde{E}(a,\lambda) + O(1/n^{2\gamma-1}) \tag{20}$$

as $n \to \infty$ and this holds uniformly with respect to $\lambda \in \Sigma$. Theorem 1.3(c) implies that $T(a-\lambda)T([a-\lambda]^{-1}) - I$ is of trace class and

$$\widetilde{E}(a,\lambda) = \det T(a-\lambda)T([a-\lambda]^{-1}) \tag{21}$$

for all $\lambda \in \Sigma$. Combining (20) and (21), we deduce that

$$\log\det T_n(a-\lambda) = (n+1)\log G(a-\lambda) + \log\det T(a-\lambda)T([a-\lambda]^{-1}) + O(1/n^{2\gamma-1})$$

as $n \to \infty$ uniformly with respect to $\lambda \in \Sigma$. Hence, one can differentiate both sides of the last formula with respect to λ, multiply by $f(\lambda)$, and integrate over $\partial\Omega$. The proof is finished by a literal repetition of Widom's proof of Theorem 1.2 (see [21, p. 21] or [5, Section 10.90]) with $o(1)$ replaced by $O(1/n^{2\gamma-1})$. □

References

[1] A. Böttcher and S. M. Grudsky, *Spectral Properties of Banded Toeplitz Operators*. SIAM, Philadelphia, PA, 2005.

[2] A. Böttcher and B. Silbermann, *Notes on the asymptotic behavior of block Toeplitz matrices and determinants*. Math. Nachr., **98** (1980), 183–210.

[3] A. Böttcher and B. Silbermann, *Invertibility and Asymptotics of Toeplitz Matrices*. Akademie-Verlag, Berlin, 1983.

[4] A. Böttcher and B. Silbermann, *Introduction to Large Truncated Toeplitz Matrices*. Springer-Verlag, New York, 1999.

[5] A. Böttcher and B. Silbermann, *Analysis of Toeplitz Operators*. 2nd edition. Springer-Verlag, Berlin, 2006.

[6] M. S. Budjanu and I. C. Gohberg, *General theorems on the factorization of matrix-valued functions. II. Some tests and their consequences*. Amer. Math. Soc. Transl. (2), **102** (1973), 15–26.

[7] K. F. Clancey and I. Gohberg, *Factorization of Matrix Functions and Singular Integral Operators*. Birkhäuser Verlag, Basel, 1981.

[8] T. Ehrhardt, *A new algebraic approach to the Szegő-Widom limit theorem*. Acta Math. Hungar., **99** (2003), 233–261.

[9] I. C. Gohberg and M. G. Krein, *Introduction to the Theory of Linear Nonselfadjoint Operators*. AMS, Providence, RI, 1969.

[10] U. Grenander and G. Szegő, *Toeplitz Forms and Their Applications*. University of California Press, Berkeley, Los Angeles, 1958.

[11] A. Yu. Karlovich, *Asymptotics of determinants and traces of Toeplitz matrices with symbols in weighted Wiener algebras*. Z. Anal. Anwendungen, **26** (2007), 43–56.

[12] M. G. Krein, *Certain new Banach algebras and theorems of the type of the Wiener-Lévy theorems for series and Fourier integrals*. Amer. Math. Soc. Transl. (2), **93** (1970), 177–199.

[13] G. S. Litvinchuk and I. M. Spitkovsky, *Factorization of Measurable Matrix Functions*. Birkhäuser Verlag, Basel, 1987.

[14] V. V. Peller, *Hankel Operators and Their Applications*. Springer, New York, 2003.

[15] S. Prössdorf and B. Silbermann, *Numerical Analysis for Integral and Related Operator Equations*. Birkhäuser Verlag, Basel, 1991.

[16] H.-J. Schmeisser and H. Triebel, *Topics in Fourier Analysis and Function Spaces*. John Wiley & Sons, Chichester, 1987.

[17] M. A. Shubin, *Factorization of matrix functions depending on a parameter in normed rings and related problems of the theory of Noetherian operators*. Matem. Sbornik, **73(115)** (1967), 610–629 (in Russian).

[18] B. Simon, *Orthogonal Polynomials on the Unit Circle. Part 1*. AMS, Providence, RI, 2005.

[19] A. F. Timan, *Theory of Approximation of Functions of a Real Variable*. Pergamon Press, Oxford, 1963.

[20] V. A. Vasil'ev, E. A. Maksimenko, and I. B. Simonenko, *On a Szegő-Widom limit theorem*. Dokl. Akad. Nauk, **393** (2003), 307–308 (in Russian).

[21] H. Widom, *Asymptotic behavior of block Toeplitz matrices and determinants. II*. Advances in Math., **21** (1976), 1–29.

Universade do Minho, Centro de Matemática, Escola de Ciências, Campus de Gualtar, 4710-057, Braga, Portugal

E-mail address: `oleksiy@math.uminho.pt`

Current address: Departamento de Matemática, Instituto Superior Técnico, Av. Rovisco Pais 1, 1049–001, Lisbon, Portugal

E-mail address: `akarlov@math.ist.utl.pt`